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Mosco-convergences of governing energies for singular vectorial systems with dynamic boundary conditions

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Introduction

Let $m \in \mathbb{N}$ and $1 < N \in \mathbb{N}$ be constants of dimensions. Let $\Omega \subset \mathbb{R}^N$ be a bounded spatial domain with a smooth boundary $\Gamma := \partial\Omega$. Also, let $\mathscr{H}$ be a product Hilbert space defined as $\mathscr{H} := L^2(\Omega; \mathbb{R}^m) \times L^2(\Gamma; \mathbb{R}^m)$.

In this paper, we study on the relationship between a convex function $\Phi_* : \mathscr{H} \to [0, \infty]$ of singular type, and a class of convex functions $\Phi_\kappa^\delta : [0, \infty] \to \mathscr{H}$, for $\kappa > 0$, $\delta \geq 0$, of regular types, defined as follows.

$$[u, u_\Gamma] \in \mathscr{W} \subset \mathscr{H} \mapsto \Phi_*(u, u_\Gamma)$$
$$:= \int_\Omega |Du| + \int_\Gamma |u_{|_\Gamma} - u_\Gamma|_{\mathbb{R}^m} \, d\Gamma + \frac{\varepsilon^2}{2} \int_\Gamma \|\nabla_\Gamma u_\Gamma\|^2 \, d\Gamma, \tag{1}$$

with the effective domain

$$\mathscr{W} := (BV(\Omega; \mathbb{R}^m) \cap L^2(\Omega; \mathbb{R}^m)) \times H^1(\Gamma; \mathbb{R}^m),$$

and for every $\kappa > 0$ and $\delta \geq 0$,

$$[u, u_\Gamma] \in \mathscr{V} \subset \mathscr{H} \mapsto \Phi_\kappa^\delta(u, u_\Gamma)$$
$$:= \int_\Omega \left(f_\delta(\nabla u) + \frac{\kappa^2}{2} \|\nabla u\|^2 \right) dx + \frac{\varepsilon^2}{2} \int_\Gamma \|\nabla_\Gamma u_\Gamma\|^2 \, d\Gamma, \tag{2}$$

with the (uniform) effective domain

$$\mathscr{V} := \left\{ [u, u_\Gamma] \in \mathscr{H} \;\middle|\; \begin{array}{l} u \in H^1(\Omega; \mathbb{R}^m),\ u_\Gamma \in H^1(\Gamma; \mathbb{R}^m) \\ \text{and } u_{|_\Gamma} = u_\Gamma \text{ on } \Gamma \end{array} \right\}$$

In the context, $\int_\Omega |Du|$ denotes the total variation of a function $u \in BV(\Omega; \mathbb{R}^m)$. $|\cdot|_{\mathbb{R}^m}$ denotes the m-dimensional Euclidean norm, and $\|\cdot\|$ denotes the Frobenius norm for $(m \times N)$-matrices. The notations "$_{|_\Gamma}$", "$d\Gamma$" and "∇_Γ" mean "the trace of a Sobolev function on Ω", "the area element on Γ" and "the surface gradient on Γ", respectively. Besides, $\{f_\delta\}_{\delta \geq 0}$ is a class of functions, consisting of the Frobenius norm $f_0 := \|\cdot\|$ and its approximating sequence $\{f_\delta\}_{\delta > 0}$, as $\delta \to 0$.

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The convex function Φ_* is a governing energy for the following system:

$$\partial_t u - \operatorname{div}\left(\frac{Du}{|Du|}\right) \ni 0 \text{ in } (0,\infty)\times\Omega, \tag{3}$$

$$\partial_t u_\Gamma - \varepsilon^2 \Delta_\Gamma u_\Gamma + \left(\tfrac{Du}{|Du|}\right)_{|_\Gamma} \boldsymbol{n}_\Gamma \ni 0 \text{ and } u_{|_\Gamma} = u_\Gamma \text{ on } (0,\infty)\times\Gamma, \tag{4}$$

which is formulated as a kind of transmission system, consisting of the vectorial singular diffusion equation (3), and the vectorial dynamic boundary condition (4). Meanwhile, for every $\kappa > 0$ and $\delta \geq 0$, the convex function Φ_κ^δ corresponds to a governing energy of the following regularized system for (3)–(4):

$$\partial_t u - \operatorname{div}\left(\partial f_\delta(\nabla u) + \kappa^2 \nabla u\right) \ni 0 \text{ in } (0,\infty)\times\Omega, \tag{5}$$

$$\partial_t u_\Gamma - \varepsilon^2 \Delta_\Gamma u_\Gamma + (\partial f_\delta(\nabla u) + \kappa^2 \nabla u)_{|_\Gamma} \boldsymbol{n}_\Gamma \ni 0 \text{ and } u_{|_\Gamma} = u_\Gamma \text{ on } (0,\infty)\times\Gamma, \tag{6}$$

consisting of the regularized diffusion equation (5), and the corresponding dynamic boundary condition (6). Here, for any $\delta \geq 0$, ∂f_δ denotes the subsidderential of f_δ.

When the unknowns u and u_Γ are scalar-valued, we can now find a relevant previous work [10], which dealt with the singular system (3)–(4). The main results of [10] were concerned with:

(a) the Mosco-convergence of the governing convex energies, under the scalar-valued settings of unknowns;

(b) the well-posedness and comparison principle for "weak-solutions";

and in this context, the weak solutions were defined on the basis of Cauchy problems of evolution equations, governed by the subdifferentials of corresponding convex energies.

As a natural consequence, it can be expected to obtain some extended results similar to [10], under the vectorial setting of unknowns. In fact, for the regularized system (5)–(6), the validity of the expectation was reported in [9], together with the precise representation results for the vectorial weak solutions.

However, if we consider the singular system (3)–(4), then we should note the gap between the mathematical treatments of the transmission condition $u_{|_\Gamma} = u_\Gamma$ as in the regular dynamic boundary condition (6), and the singular one (4). More precisely, the transmission condition works as a functional constraint in the definition (2) of regular energies Φ_κ^δ, but it does not work in the definition (1) of singular energy Φ_*. This gap brings us a question to ask the rigorous mathematical expression of the transmission condition, which is replaced by the weak solutions to the singular system (3)–(4).

The previous result (a) of Mosco-convergence will provide an important clue to address this question, and the generalization approach in vectorial frameworks will lead to the enhancement of mathematical theory that enables us to handle various singular situations, as in Bingham type flow, Ginzburg–Landau type equations, and so on.

In view of these, we set the goal of this paper to prove the following Main Theorem, that corresponds to the generalization for the previous result (a).

Main Theorem 1. To conclude the Mosco-convergence $\Phi_n := \Phi_{\kappa_n}^{\delta_n} \to \Phi_*$ on $\mathscr{H}$, for any limiting sequence:

$$0 \leq \delta_n \to 0 \text{ and } 0 < \kappa_n \to 0, \text{ as } n \to \infty,$$

under the vectorial setting of the unknowns.

In this paper, the discussion for Main Theorem 1 is developed in accordance with the following contents. In Section 1, we prepare preliminaries of this study, containing the notations for the treatments of vectorial functions. On this basis of Section 1, we state Main Theorem 1 in Section 2 and the Key-Lemma A and B to fill the above-mentioned the gaps. The results are proved through the following Section 3. Finally, Section 4 is given the proof of Main Theorem 1, based on the preceding Sections.

1 Preliminaries

In this section, we outline some basic notations, as preliminaries of our study.

Notation 1 (Notations in real analysis). For arbitrary $\alpha, \beta \in [-\infty, \infty]$, we define:

$$\alpha \vee \beta := \max\{\alpha, \beta\} \text{ and } \alpha \wedge \beta := \min\{\alpha, \beta\};$$

and in particular, we write $[\alpha]^+ := \alpha \vee 0$ and $[\beta]^- := -(0 \wedge \beta)$.

Let $d \in \mathbb{N}$ be any fixed dimension. Then, we simply denote by $a \cdot b$ and $|a|_{\mathbb{R}^d}$ the standard scalar product of $a, b \in \mathbb{R}^d$ and the d-dimensional Euclidean norm of $a \in \mathbb{R}^d$, respectively. Also, we denote by

$$\mathbb{B}^d := \{a \in \mathbb{R}^d \,|\, |a|_{\mathbb{R}^d} < 1\} \text{ and } \mathbb{S}^{d-1} := \{a \in \mathbb{R}^d \,|\, |a|_{\mathbb{R}^d} = 1\}$$

the d-dimensional unit open ball centered at the origin, and its boundary, respectively. In particular, when $d > 1$, we set:

$$[a]^+ := \bigl[[a_1]^+, \ldots, [a_d]^+\bigr] \text{ and } [b]^- := \bigl[[b_1]^-, \ldots, [b_m]^-\bigr], \text{ for all } a, b \in \mathbb{R}^d.$$

Besides, we often describe a d-dimensional vector $a = [a_1, \ldots, a_d] \in \mathbb{R}^d$ as $a = [\tilde{a}, a_d]$ by putting $\tilde{a} = [a_1, \ldots, a_{d-1}] \in \mathbb{R}^{d-1}$. As well as, we describe the gradient $\nabla = [\partial_1, \ldots \partial_d]$ as $\nabla = [\tilde{\nabla}, \partial_d]$ by putting $\tilde{\nabla} = [\partial_1, \ldots, \partial_{d-1}]$, and we describe $\nabla_x, \partial_t, \partial_{x_d}$, and so on, when we need to specify the variables of differentials. For every two vectors a, $b \in \mathbb{R}^d$, we denote by $a \otimes b$ the tensor product of a and b, i.e.:

$$a \otimes b := a^t b = \begin{bmatrix} a_1 b_1 & \cdots & a_1 b_d \\ \vdots & \ddots & \vdots \\ a_d b_1 & \cdots & a_d b_d \end{bmatrix} \in \mathbb{R}^{d \times d}.$$

Additionally, let $m \in \mathbb{N}$ be another dimension (besides d) in this paper. For arbitrary $(m \times d)$-matrices $A = [a_{ij}], B = [b_{ij}] \in \mathbb{R}^{m \times d}$ with components $a_{ij}, b_{ij} \in \mathbb{R}$ ($i = 1, \ldots m$, $j = 1, \ldots, d$), we denote by $A : B$ and $\|A\|$ the scalar product of A and B and the Frobenius norm of A, respectively, i.e.:

$$A : B := \sum_{j=1}^{d} \sum_{i=1}^{m} a_{ij} b_{ij} \in \mathbb{R} \text{ and } \|A\| := \sqrt{A : A} \in \mathbb{R}, \text{ for all } A, B \in \mathbb{R}^{m \times d}.$$

For any $d \in \mathbb{N}$, the d-dimensional Lebesgue measure is denoted by $\mathcal{L}^d$, and unless otherwise specified, the measure theoretical phrases, such as "a.e.", "dt", "dx", and so on, are with respect to the Lebesgue measure in each corresponding dimension. Also, in the observations on a C^1-surface S, the phrase "a.e." is with respect to the Hausdorff measure in each corresponding Hausdorff dimension, and the area element on S is denoted by dS.

Notation 2 (Notations of functional analysis)**.** For an abstract Banach space X, we denote by $|\cdot|_X$ the norm of X, and denote by $\langle \cdot, \cdot \rangle_X$ the duality pairing between X and the dual space X^* of X. In particular, when X is a Hilbert space, we denote by $(\cdot, \cdot)_X$ the inner product in X.

Notation 3 (Notations in convex analysis)**.** Let X be an abstract real Hilbert space. For any proper lower semi-continuous (l.s.c. from now on) and convex function Ψ defined on X, we denote by $D(\Psi)$ its effective domain, and denote by $\partial\Psi$ its subdifferential. The subdifferential $\partial\Psi$ is a set-valued map corresponding to a weak differential of Ψ, and it has a maximal monotone graph in the product space $X \times X$. More precisely, for each $z_0 \in X$, the value $\partial\Psi(z_0)$ is defined as a set of all elements $z_0^* \in X$ which satisfy the following variational inequality:

$$(z_0^*, z - z_0)_X \leq \Psi(z) - \Psi(z_0), \text{ for any } z \in D(\Psi).$$

The set $D(\partial\Psi) := \{z \in X \,|\, \partial\Psi(z) \neq \emptyset\}$ is called the domain of $\partial\Psi$, and it is often said "$[z_0, z_0^*] \in \partial\Psi$ in $X \times X$" to mean "$z_0 \in D(\partial\Psi)$ and $z_0^* \in \partial\Psi(z_0)$ in X" by identifying the operator $\partial\Psi$ with its graph in $X \times X$.

On this basis, we recall the notion of "Mosco-convergence" for sequences of convex functions.

Definition 1.1 (Mosco-convergence: cf. [8])**.** Let X be an abstract Hilbert space. Let $\Psi : X \to (-\infty, \infty]$ be a proper l.s.c. and convex function, and let $\{\Psi_n\}_{n=1}^{\infty}$ be a sequence of proper l.s.c. and convex functions $\Psi_n : X \to (-\infty, \infty]$, $n \in \mathbb{N}$. Then, it is said that $\Psi_n \to \Psi$ on X, in the sense of Mosco, as $n \to \infty$, iff. the following two conditions are fulfilled.

(M1) Lower-bound: $\varliminf_{n\to\infty} \Psi_n(\check{z}_n) \geq \Psi(\check{z})$, if $\check{z} \in X$, $\{\check{z}_n\}_{n=1}^{\infty} \subset X$, and $\check{z}_n \to \check{z}$ weakly in X as $n \to \infty$.

(M2) Optimality: for any $\hat{z} \in D(\Psi)$, there exists a sequence $\{\hat{z}_n\}_{n=1}^{\infty} \subset X$ such that $\hat{z}_n \to \hat{z}$ in X and $\Psi_n(\hat{z}_n) \to \Psi(\hat{z})$, as $n \to \infty$.

Next, we prepare the notations associated with the spatial domain Ω and those based on the settings of this domain.

Notation 4 (Notations for the spatial domain)**.** Throughout this paper, let $1 < N \in \mathbb{N}$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^∞-boundary $\Gamma := \partial\Omega$ and the unit outer normal $\boldsymbol{n}_\Gamma \in C^\infty(\Gamma; \mathbb{R}^N)$. Besides, we suppose that Ω and Γ fulfill the following two conditions.

(Ω0) There exists a small constant $r_\Gamma > 0$, and the mapping

$$d_\Gamma : x \in \overline{\Omega} \mapsto \inf_{y \in \Gamma} |x - y| \in [0, \infty)$$

forms a smooth function on the neighborhoods of Γ:

$$\Gamma(r) := \left\{ \ x \in \Omega \ \middle| \ d_\Gamma(x) < r \ \right\}, \text{ for every } r \in (0, r_\Gamma].$$

(Ω1) There exists a small constant $r_* \in (0, r_\Gamma]$, and for any $x_\Gamma \in \Gamma$ and arbitrary $\rho, r \in (0, r_*]$, the neighborhood:

$$G_{x_\Gamma}(\rho, r) := \left\{ \ y + x_\Gamma + \tau \boldsymbol{n}_\Gamma \ \middle| \ \begin{array}{l} \tau \in (-r, r),\ y \in \Gamma - x_\Gamma, \text{ and} \\ \left| y - (y \cdot \boldsymbol{n}_\Gamma(x_\Gamma)) \boldsymbol{n}_\Gamma(x_\Gamma) \right| < \rho \end{array} \right\},$$

is transformed to a cylinder:

$$\Pi_0(\rho, r) := \left\{ \ \xi = [\tilde{\xi}, \xi_N] \in \mathbb{R}^N \ \middle| \ \tilde{\xi} \in \rho \mathbb{B}^{N-1} \text{ and } \xi_N \in (-r, r) \ \right\},$$

by using a uniform C^∞-diffeomorphism $\Xi_{x_\Gamma} : G_{x_\Gamma}(r_*, r_*) \to \Pi_0(r_*, r_*)$. Additionally, for any $x_\Gamma \in \Gamma$, there exists a function $\gamma_{x_\Gamma} \in C^\infty(r_* \overline{\mathbb{B}^{N-1}})$, a congruence transform $\Lambda_{x_\Gamma} : \mathbb{R}^N \to \mathbb{R}^N$ and a C^∞-diffeomorphism $H_{x_\Gamma} : \Lambda_{x_\Gamma} G_{x_\Gamma}(r_*, r_*) \to \Pi_0(r_*, r_*)$ such that:

(ω0) $\Xi_{x_\Gamma} = H_{x_\Gamma} \circ \Lambda_{x_\Gamma}$ as a mapping from $G_{x_\Gamma}(r_*, r_*)$ onto $\Pi_0(r_*, r_*)$;

(ω1) $\gamma_{x_\Gamma}(0) = 0$, and $\nabla \gamma_{x_\Gamma}(0) = 0$ in $\mathbb{R}^{N-1}$;

(ω2) for every $\rho, r \in (0, r_*]$,

$$\Lambda_{x_\Gamma} G_{x_\Gamma}(\rho, r) = Y_{x_\Gamma}(\rho, r) := \left\{ \ y = [\tilde{y}, y_N] \in \mathbb{R}^N \ \middle| \ [\tilde{y}, y_N - \gamma_{x_\Gamma}(\tilde{y})] \in \Pi_0(\rho, r) \ \right\},$$

and in particular,

$$\Lambda_{x_\Gamma} \big(\Gamma \cap G_{x_\Gamma}(\rho, r) \big) = \left\{ \ y = [\tilde{y}, \gamma_{x_\Gamma}(\tilde{y})] \in \mathbb{R}^N \ \middle| \ \tilde{y} \in \rho \mathbb{B}^{N-1} \ \right\};$$

(ω3) for every $\rho, r \in (0, r_*]$,

$$H_{x_\Gamma} : y = [\tilde{y}, y_N] \in Y_{x_\Gamma}(\rho, r) \mapsto \xi = H_{x_\Gamma} y := [\tilde{y}, y_N - \gamma_{x_\Gamma}(\tilde{y})] \in \Pi_0(\rho, r).$$

Remark 1.1. From (Ω0), we may further suppose the following condition.

(Ω2) For any $\sigma > 0$, there exists a constant $\rho_*^\sigma \in (0, r_*]$ such that:

$$\rho_*^\sigma \leq \sigma,\ |\gamma_{x_\Gamma}|_{C^1(\rho \overline{\mathbb{B}^{N-1}})} \leq \sigma \text{ and } \left\{ \ \Xi_{x_\Gamma}^{-1}[\tilde{\xi}, \gamma_{x_\Gamma}(\tilde{\xi}) + r_*] \ \middle| \ \tilde{\xi} \in \rho \overline{\mathbb{B}^{N-1}} \ \right\} \cap \overline{\Gamma(r_*/2)} = \emptyset,$$
$$\text{for any } x_\Gamma \in \Gamma \text{ and any } \rho \in (0, \rho_*^\sigma].$$

Notation 5 (Notations in BV-theory: cf. [1, 4–6]). Let $1 < N \in \mathbb{N}$, $m \in \mathbb{N}$ be fixed constants of dimensions, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\Gamma := \partial\Omega$ as in Notation 4. Then, we denote by $\mathcal{M}(\Omega)^m$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega)^m$) the space of all finite $\mathbb{R}^m$-valued Radon measures (resp. the space of all $\mathbb{R}^m$-valued Radon measures) on Ω.

In general, the space $\mathcal{M}(\Omega)^m$ (resp. $\mathcal{M}_{\mathrm{loc}}(\Omega)^m$) is known as the dual of the Banach space $C_0(\Omega;\mathbb{R}^m)$ (resp. dual of the locally convex space $C_{\mathrm{c}}(\Omega;\mathbb{R}^m)$).

A function $z \in L^1(\Omega;\mathbb{R}^m)$ (resp. $z \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$) is called a function of bounded variation, or a BV-function (resp. a function of locally bounded variation, or a $\mathrm{BV}_{\mathrm{loc}}$-function) on Ω, iff. its distributional differential Dz is a finite $\mathbb{R}^{m\times N}$-valued Radon measure on Ω (resp. a $\mathbb{R}^{m\times N}$-valued Radon measure on Ω), namely $Du \in \mathcal{M}(\Omega)^{m\times N}$ (resp. $Du \in \mathcal{M}_{\mathrm{loc}}(\Omega)^{m\times N}$).

We denote by $BV(\Omega;\mathbb{R}^m)$ (resp. $BV_{\mathrm{loc}}(\Omega;\mathbb{R}^m)$) the space of all BV-functions (resp. all $\mathrm{BV}_{\mathrm{loc}}$-functions) on Ω. For any $z \in BV(\Omega;\mathbb{R}^m)$, the Radon measure Dz is called the variation measure of z, and its total variation $|Dz|$ is called the total variation measure of z. Additionally, the value $|Dz|(\Omega)$, for any $z \in BV(\Omega;\mathbb{R}^m)$, can be calculated as follows:

$$|Dz|(\Omega) = \sup\left\{ \int_\Omega z \cdot \operatorname{div}\Phi\, dx \;\middle|\; \Phi \in C^1_{\mathrm{c}}(\Omega;\mathbb{R}^{m\times N}) \text{ and } \|\Phi\| \le 1 \text{ on } \Omega \right\}.$$

The space $BV(\Omega;\mathbb{R}^m)$ is a Banach space, endowed with the following norm:

$$|z|_{BV(\Omega;\mathbb{R}^m)} := |z|_{L^1(\Omega;\mathbb{R}^m)} + |Dz|(\Omega), \quad \text{for any } z \in BV(\Omega;\mathbb{R}^m).$$

Also, $BV(\Omega;\mathbb{R}^m)$ is a metric space, endowed with the following distance:

$$[z,w] \in BV(\Omega;\mathbb{R}^m)^2 \mapsto |z-w|_{L^1(\Omega;\mathbb{R}^m)} + \left| \int_\Omega |Dz| - \int_\Omega |Dw| \right|.$$

The topology provided by this distance is called the *strict topology* of $BV(\Omega;\mathbb{R}^m)$ and the convergence of sequence in the strict topology is often phrased as "strictly in $BV(\Omega;\mathbb{R}^m)$".

In the meantime, there exists a (unique) bounded linear operator $\mathcal{T}_\Gamma : BV(\Omega;\mathbb{R}^m) \mapsto L^1(\Gamma;\mathbb{R}^m)$, called *trace* such that $\mathcal{T}_\Gamma\varphi = \varphi|_\Gamma$ on Γ for any $\varphi \in C^1(\overline{\Omega};\mathbb{R}^m)$. Hence, in this paper, we shortly denote the value of trace $\mathcal{T}_\Gamma z \in L^1(\Gamma;\mathbb{R}^m)$ by $z_{|_\Gamma}$. Additionally, if $1 \le r < \infty$, then the space $C^\infty(\overline{\Omega};\mathbb{R}^m)$ is dense in $BV(\Omega;\mathbb{R}^m) \cap L^r(\Omega;\mathbb{R}^m)$ for the *intermediate convergence* (cf. [4, Definition 10.1.3. and Theorem 10.1.2]), i.e. for any $z \in BV(\Omega;\mathbb{R}^m) \cap L^r(\Omega;\mathbb{R}^m)$, there exists a sequence $\{z_n\}_{n=1}^\infty \subset C^\infty(\overline{\Omega})$ such that $z_n \to z$ in $L^r(\Omega;\mathbb{R}^m)$ and $\int_\Omega \|\nabla z_n\| dx \to |Dz|(\Omega)$ as $n \to \infty$.

Remark 1.2. (cf. [1, Theorem 3.88]) Let $\mathcal{T}_\Gamma : BV(\Omega;\mathbb{R}^m) \to L^1(\Gamma;\mathbb{R}^m)$ be the trace for the vectorial functions. Then, it holds that:

$$\int_\Gamma z_{|_\Gamma} \cdot (\Psi \boldsymbol{n}_\Gamma)\, d\Gamma = \int_\Omega z \cdot \operatorname{div}\Psi\, dx + \int_\Omega \Psi : Dz, \text{ for any } \Psi \in C^1_{\mathrm{c}}(\mathbb{R}^m;\mathbb{R}^{m\times N}),$$

Moreover, the trace $\mathcal{T}_\Gamma$ is continuous with respect to the strict topology of $BV(\Omega;\mathbb{R}^m)$. Namely, the convergence of continuous dependence holds:

$$\mathcal{T}_\Gamma z_n \to \mathcal{T}_\Gamma z \text{ as } n \to \infty, \text{ for } z \in BV(\Omega;\mathbb{R}^m) \text{ and } \{z_n\}_{n=1}^\infty \subset BV(\Omega;\mathbb{R}^m),$$

in the topology of $L^1(\Gamma;\mathbb{R}^m)$, if $z_n \to z$ strictly in $BV(\Omega;\mathbb{R}^m)$. However, in contrast with the traces on Sobolev spaces, it must be noted that the convergence is not guaranteed, if $z_n \to z$ weakly-$*$ in $BV(\Omega;\mathbb{R}^m)$, and even if we adopt any weak topology for the above convergence (including the distributional one).

Notation 6 (Extensions of functions: cf. [1,4]). Let μ be a positive measure on $\mathbb{R}^N$, and let $B \subset \mathbb{R}^N$ be a μ-measurable Borel set. For any μ-measurable function $u : B \to \mathbb{R}^m$, we denote by $[u]^{\mathrm{ex}}$ an extension of u over $\mathbb{R}^N$. More precisely, $[u]^{\mathrm{ex}} : \mathbb{R}^N \to \mathbb{R}^m$ is a Lebesgue measurable function such that $[u]^{\mathrm{ex}}$ has an expression as a μ-measurable function on B, and $[u]^{\mathrm{ex}} = u$, μ-a.e. in B. In general, the extension of $[u]^{\mathrm{ex}} : \mathbb{R}^N \to \mathbb{R}^m$ is not unique, for each $u : B \to \mathbb{R}^m$.

Remark 1.3. Let $1 < N \in \mathbb{N}$, and let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a C^1-boundary Γ. Then, for the extensions of functions in $BV(\Omega; \mathbb{R}^m)$ and $H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$, we can check the following facts.

(Fact 1) (cf. [1, Proposition 3.21]) There exists a bounded linear operator $\mathcal{E}_\Omega : BV(\Omega; \mathbb{R}^m) \to BV(\mathbb{R}^N; \mathbb{R}^m)$, such that:

- $\mathcal{E}_\Omega$ maps any function $u \in BV(\Omega; \mathbb{R}^m)$ to an extension $[u]^{\mathrm{ex}} \in BV(\mathbb{R}^N; \mathbb{R}^m)$;
- for any $1 \leq q < \infty$, $\mathcal{E}_\Omega(W^{1,q}(\Omega; \mathbb{R}^m)) \subset W^{1,q}(\mathbb{R}^N; \mathbb{R}^m)$, and the restriction $\mathcal{E}_\Omega|_{W^{1,q}(\Omega;\mathbb{R}^m)} : W^{1,q}(\Omega; \mathbb{R}^m) \to W^{1,q}(\mathbb{R}^N; \mathbb{R}^m)$ forms a bounded and linear operator with respect to the (strong-) topologies of the restricted Sobolev spaces.

(Fact 2) (cf. [4, Theorem 5.4.1 and Proposition 5.6.3]) There exists a bounded linear operator $\mathcal{E}_\Gamma : H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m) \to H^1(\mathbb{R}^N; \mathbb{R}^m)$, which maps any function $\varrho \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$ to an extension $[\varrho]^{\mathrm{ex}} \in H^1(\mathbb{R}^N; \mathbb{R}^m)$.

Based on this, we state the notations of surface-differentials.

Notation 7 (Notations of surface-differentials). Under the assumption (Ω0) in Notation 4, we can put:

$$L^2_{\mathrm{tan}}(\Gamma) := \left\{ \tilde{\omega} \in L^2(\Gamma; \mathbb{R}^N) \,\middle|\, \tilde{\omega} \cdot \boldsymbol{n}_\Gamma = 0, \text{ a.e. on } \Gamma \right\},$$

and define the Laplacian Δ_Γ on the surface Γ, i.e. the so-called *Laplace–Beltrami operator*, as the composition $\Delta_\Gamma := \mathrm{div}_\Gamma \circ \nabla_\Gamma : C^\infty(\Gamma) \to C^\infty(\Gamma)$ of the *surface gradient:*

$$\nabla_\Gamma \varphi := \nabla[\varphi]^{\mathrm{ex}} - (\nabla d_\Gamma \otimes \nabla d_\Gamma)\nabla[\varphi]^{\mathrm{ex}},$$

and the *surface-divergence:*

$$\mathrm{div}_\Gamma\, \omega := \mathrm{div}[\omega]^{\mathrm{ex}} - \nabla([\omega]^{\mathrm{ex}} \cdot \nabla d_\Gamma) \cdot \nabla d_\Gamma.$$

As is well-known (cf. [11]), the values $\nabla_\Gamma \varphi$ and $\mathrm{div}_\Gamma\, \omega$ are determined independently with respect to the choices of the extensions $\varphi^{\mathrm{ex}} \in C^\infty(\mathbb{R}^N)$ and $\omega^{\mathrm{ex}} \in C^\infty(\mathbb{R}; \mathbb{R}^N)$, and also, the operator $-\Delta_\Gamma$ can be extended to a duality map between $H^1(\Gamma)$ and $H^{-1}(\Gamma)$, via the following variational identity:

$$\langle -\Delta_\Gamma \varphi, \psi \rangle_{H^1(\Gamma)} = (\nabla_\Gamma \varphi, \nabla_\Gamma \psi)_{L^2(\Gamma;\mathbb{R}^N)}, \text{ for all } [\varphi, \psi] \in H^1(\Gamma)^2.$$

Finally, we here prepare the notations concerned with the tensor analysis.

Notation 8 (Notations in tensor analysis). In this paper, from now on, we denote by ∇z the (distributional) gradient of any vectorial function $z = [z_i] \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^m)$, defined as:

$$\nabla z := {}^{\mathrm{t}}[\nabla z_1, \dots, \nabla z_m] = \begin{bmatrix} \partial_1 z_1 & \cdots & \partial_N z_1 \\ \vdots & \ddots & \vdots \\ \partial_1 z_m & \cdots & \partial_N z_m \end{bmatrix} \in \mathcal{D}'(\Omega)^{m \times N},$$

and, we denote by $\operatorname{div} Z$ the (distributional) divergence of any matrix-valued function $Z = [z_{ij}] \in L^1_{\mathrm{loc}}(\Omega; \mathbb{R}^{m \times N})$, defined as:

$$\operatorname{div} Z := \left[\sum_{j=1}^{N} \partial_j z_{ij} \right] \in \mathcal{D}'(\Omega)^m.$$

Similarly, for any vectorial function $z = [z_i] \in H^1(\Gamma; \mathbb{R}^m)$, we define the surface-gradient $\nabla_\Gamma z$ of z by $\nabla_\Gamma z := {}^{\mathrm{t}}[\nabla_\Gamma z_1, \dots, \nabla_\Gamma z_m] \in L^2_{\mathrm{tan}}(\Gamma)^m$, and we define $\Delta_\Gamma z := [\Delta_\Gamma z_i] \in H^{-1}(\Gamma; \mathbb{R}^m)$.

Finally, we prescribe other specific notations.

Notation 9. Let $R_\Omega > 0$ be a sufficiently large constant, such that $\mathbb{B}_\Omega := R_\Omega \mathbb{B}^N \supset \overline{\Omega}$. Besides, for any $u \in BV(\Omega; \mathbb{R}^m)$ and any $g \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$, we denote by $[u]^{\mathrm{ex}}_g \in BV_{\mathrm{loc}}(\mathbb{R}^N; \mathbb{R}^m) \cap BV(\mathbb{B}_\Omega; \mathbb{R}^m) \cap H^1(\mathbb{B}_\Omega \setminus \overline{\Omega}; \mathbb{R}^m)$ an extension of u, provided as:

$$x \in \mathbb{R}^N \mapsto [u]^{\mathrm{ex}}_g(x) := \begin{cases} u(x), \text{ if } x \in \Omega, \\ [g]^{\mathrm{ex}}(x), \text{ if } x \in \mathbb{B}_\Omega \setminus \overline{\Omega}, \end{cases}$$

with the use of an extension $[g]^{\mathrm{ex}} \in H^1(\mathbb{R}^N; \mathbb{R}^m)$ of g.

2 Main Theorem

We begin with specifying the assumptions in our study.

(A0) $\varepsilon > 0$ is a fixed constant, $\kappa > 0$ and $\delta \geq 0$ are given constants. Also, $1 < N \in \mathbb{N}$, $m \in \mathbb{N}$ are fixed constants of dimensions. Ω is a bounded spatial domain in $\mathbb{R}^N$ with a smooth boundary $\Gamma := \partial\Omega$, and the unit outer normal to Γ, that fulfills the conditions (Ω0)–(Ω1) in Notation 4.

(A1) $\{f_\delta\}_{\delta > 0} \subset W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^{m \times N})$ is a class of convex functions fulfilling the following items:

(a0) $f_0 := \|\cdot\|$ on $\mathbb{R}^{m \times N}$, and for any $\delta > 0$, $0 \leq f_\delta \in C^1(\mathbb{R}^{m \times N})$ is a convex function such that $f_\delta(O) = 0$;

(a1) there exist constants $C_k > 0$, for $k = 0, 1, 2$, such that

$$\begin{cases} f_\delta(W) \geq \|W\| - \delta C_0, \\ \|\nabla f_\delta(W)\| \leq C_1 \|W\| + C_2, \end{cases} \quad \text{for any } \delta > 0 \text{ and } W \in \mathbb{R}^{m \times N};$$

(a2) for any $W \in \mathbb{R}^{m \times N}$, $f_\delta(W) \to \|W\|$ as $\delta \to 0$.

Let $\mathscr{H} := L^2(\Omega; \mathbb{R}^m) \times L^2(\Gamma; \mathbb{R}^m)$ be the product Hilbert space defined in Introduction. Also, let $\mathscr{W}$ and $\mathscr{V}$ be the subspace of $\mathscr{H}$, that are respectively given in (1) and (2) as the effective domain of the singular convex function Φ_* and the regular ones Φ_κ^δ, for $\kappa > 0$ and $\delta \geq 0$.

Now, the Main Theorem of this paper is stated as follows.

Main Theorem 1 (Mosco-convergence for convex energies). *Let $\Phi_* : \mathscr{H} \to [0, \infty]$ be the functional, given in (1), and for every $\kappa > 0$ and $\delta \geq 0$, let $\Phi_\kappa^\delta : \mathscr{H} \to [0, \infty]$ be the convex function, given in (2). Let $\{\kappa_n\}_{n=1}^\infty \subset (0, \infty)$, $\{\delta_n\}_{n=1}^\infty \subset [0, \infty)$ be arbitrary sequences, such that:*

$$\kappa_n \to 0 \text{ and } \delta_n \to 0, \text{ as } n \to \infty. \tag{2.1}$$

Then, it holds that:

$$\Phi_n := \Phi_{\kappa_n}^{\delta_n} \to \Phi_* \text{ on } \mathscr{H}, \text{ in the sense of Mosco, as } n \to \infty.$$

The proof of the Main Theorem 1 will be based on the following Key-Lemmas and Remarks.

Key-Lemma A (Key-property of Φ_*). The functional $\Phi_* : \mathscr{H} \to [0, \infty)$, given in (1), is proper l.s.c. and convex function on $\mathscr{H}$.

Remark 2.1. Key-properties of Φ_κ^δ, for $\kappa > 0$ and $\delta \geq 0$, were verified in [9]. So, we can now say that the functional Φ_κ^δ, for every $\kappa > 0$ and $\delta \geq 0$, is proper l.s.c. and convex function on $\mathscr{H}$.

Key-Lemma B (Approximating sequences for vectorial BV-functions). For any $\hat{W} = [\hat{w}, \hat{w}_\Gamma]$, there exists a sequence $\{\hat{w}_\ell\}_{\ell=1}^\infty \subset H^1(\Gamma; \mathbb{R}^m)$, such that:

$$\hat{w}_{|\Gamma} = \hat{w}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m), \text{ for any } \ell \in \mathbb{N},$$

and

$$\begin{cases} \hat{w}_\ell \to \hat{w} \text{ in } L^2(\Omega; \mathbb{R}^m), \\ \displaystyle\int_\Omega \|\nabla \hat{w}_\ell\| \, dx \to \int_\Omega |D\hat{w}| + \int_\Gamma |\hat{w}_{|\Gamma} - \hat{w}_\Gamma|_{\mathbb{R}^m} \, d\Gamma, \end{cases} \text{as } \ell \to \infty.$$

3 Proofs of Key-Lemmas

In this section, we show the Key-Lemmas in the preceding section.

Lemma A1. *For any $v \in BV(\Omega; \mathbb{R}^m)$ and any $g \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$, let $[v]_g^{\mathrm{ex}}$ be the extension of v, defined in Notation 9. Then, $[v]_g^{\mathrm{ex}}$ belongs to $BV(\mathbb{B}_\Omega; \mathbb{R}^m)$ and it holds that:*

$$D[v]_g^{\mathrm{ex}} = Dv + \nabla[g]^{\mathrm{ex}} \mathcal{L}^N \lfloor_{(\mathbb{B}_\Omega \setminus \overline{\Omega})} + (v_{|\Gamma} - g) \otimes (-\boldsymbol{n}_\Gamma) \mathcal{H}^{N-1} \lfloor_\Gamma \text{ in } \mathcal{M}(\mathbb{B}_\Omega)^{m \times N}, \tag{3.1}$$

and therefore,

$$|D[v]_g^{\mathrm{ex}}| = |Dv| + \|\nabla[g]^{\mathrm{ex}}\| \lfloor_{(\mathbb{B}_\Omega \setminus \overline{\Omega})} + |v_{|\Gamma} - g|_{\mathbb{R}^m} \mathcal{H}^{N-1} \lfloor_\Gamma \text{ in } \mathcal{M}(\mathbb{B}_\Omega). \tag{3.2}$$

Proof. The proof of Lemma A1 will be directly obtained by applying the general theories (cf. [1, Theorem 3.84 and Corollary 3.89], [4, Example 10.2.1] and [5, Theorem 5.8]). However, we report the proof for the reader's convenience.

Let us fix any Borel set $B \subset \mathbb{B}_\Omega$, and let us take any function $\Psi \in C^1_c(\mathbb{B}_\Omega; \mathbb{R}^{m\times N})$, satisfying $\|\Psi\| \le 1$ on $\mathbb{B}_\Omega$. From Remarks 1.2–1.3, it can be seen that:

$$\begin{aligned} \int_B [v]^{\mathrm{ex}}_g \cdot \operatorname{div} \Psi \, dx &= \int_{B\cap\Omega} v \cdot \operatorname{div} \Psi \, dx + \int_{B\setminus\overline{\Omega}} [g]^{\mathrm{ex}} \cdot \operatorname{div} \Psi \, dx \\ &= -\int_{B\cap\Omega} \Psi : Dv - \int_{B\setminus\overline{\Omega}} \Psi : \nabla[g]^{\mathrm{ex}} \, dx + \int_{B\cap\Gamma} (v_{|\Gamma} - g) \cdot (\Psi \boldsymbol{n}_\Gamma) \, d\Gamma \\ &\le \int_{B\cap\Omega} |Dv| + \int_{B\setminus\overline{\Omega}} \|\nabla[g]^{\mathrm{ex}}\| \, dx + \int_{B\cap\Gamma} |v_{|\Gamma} - g|_{\mathbb{R}^m} \, d\Gamma. \end{aligned}$$

The above calculation implies that:

$$|D[v]^{\mathrm{ex}}_g|(B) \le \int_{B\cap\Omega} |Dv| + \int_{B\setminus\overline{\Omega}} \|\nabla[g]^{\mathrm{ex}}\| \, dx + \int_{B\cap\Gamma} |v_{|\Gamma} - g|_{\mathbb{R}^m} \, d\Gamma, \tag{3.3}$$

and

$$[v]^{\mathrm{ex}}_g \in BV(\mathbb{B}_\Omega; \mathbb{R}^m).$$

Next, we invoke [1, Theorem 3.84] to observe that:

$$\begin{aligned} \int_{\mathbb{B}_\Omega} \tilde{\Psi} : D[v]^{\mathrm{ex}}_g = & \int_\Omega \tilde{\Psi} : Dv + \int_{\mathbb{B}_\Omega\setminus\overline{\Omega}} \nabla[g]^{\mathrm{ex}} : \tilde{\Psi} \, dx \\ & + \int_\Gamma (v_{|\Gamma} - g) \otimes (-\boldsymbol{n}_\Gamma) : \tilde{\Psi} \, d\Gamma, \text{ for any } \tilde{\Psi} \in C_c(\mathbb{B}_\Omega; \mathbb{R}^m). \end{aligned} \tag{3.4}$$

By this identity, we immediately have:

$$\begin{cases} D[v]^{\mathrm{ex}}_g \lfloor_\Omega = Dv \text{ in } \mathcal{M}(\Omega)^{m\times N}, \\ D[v]^{\mathrm{ex}}_g \lfloor_{\mathbb{B}_\Omega\setminus\overline{\Omega}} = \nabla[g]^{\mathrm{ex}} \mathcal{L}^N \text{ in } \mathcal{M}(\mathbb{B}_\Omega \setminus \overline{\Omega})^{m\times N}. \end{cases} \tag{3.5}$$

Subsequently, from (3.3)–(3.4), it can be seen that:

$$D[v]^{\mathrm{ex}}_g \lfloor_\Gamma = (v_{|\Gamma} - g) \otimes (-\boldsymbol{n}_\Gamma) \mathcal{H}^{N-1} \text{ in } \mathcal{M}(\Gamma)^{m\times N}. \tag{3.6}$$

(3.5)–(3.6) imply (3.1)–(3.2). □

Proof of Key-Lemma A. From (1), the definition of Φ_*, we immediately see that Φ_* is proper and convex. So, we here verify only the lower semi-continuity of Φ_*.

Let us fix any $g \in H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m)$. Then, by the preceding lemma, the functional:

$$\begin{aligned} v \in\ & L^1(\Omega; \mathbb{R}^m) \mapsto |D[v]^{\mathrm{ex}}_g|(\overline{\Omega}) \\ := & \begin{cases} \displaystyle\int_\Omega |Dv| + \int_\Gamma |v_{|\Gamma} - g|_{\mathbb{R}^m} \, d\Gamma = |D[v]^{\mathrm{ex}}_g|(\mathbb{B}_\Omega) - |D[g]^{\mathrm{ex}}|(\mathbb{B}_\Omega \setminus \overline{\Omega}), \\ \qquad \text{if } v \in BV(\Omega; \mathbb{R}^m), \\ \infty, \quad \text{otherwise}, \end{cases} \end{aligned} \tag{3.7}$$

forms a proper l.s.c. and convex function on $L^1(\Omega;\mathbb{R}^m)$. Moreover, invoking Remark 1.2, it can be seen that:

$$|D[v_n]_g^{\mathrm{ex}}|(\overline{\Omega}) \to |D[v]_g^{\mathrm{ex}}|(\overline{\Omega}), \text{ as } n \to \infty, \tag{3.8}$$

whenever $\{v_n\}_{n=1}^\infty \subset BV(\Omega;\mathbb{R}^m)\cap L^2(\Omega;\mathbb{R}^m)$, $v \in BV(\Omega;\mathbb{R}^m)\cap L^2(\Omega;\mathbb{R}^m)$ and $v_n \to v$ in $L^2(\Omega;\mathbb{R}^m)$ and strictly in $BV(\Omega;\mathbb{R}^m)$, as $n\to\infty$.

On this basis, we fix $W=[w,w_\Gamma]$ and take any sequence $\{W_n=[w_n,w_{\Gamma,n}]\}_{n=1}^\infty \subset \mathscr{W}$, such that $\{W_n\}_{n=1}^\infty$ converges to W in the topology of $\mathscr{H}$. Then, on account of (3.7)–(3.8), the lower semi-continuity of Φ_* is verified as follows.

$$\begin{aligned}
\varliminf_{n\to\infty} \Phi_*(W_n) &\geq \varliminf_{n\to\infty} \left(\int_\Omega |Dw_n| + \int_\Gamma |w_{n|\Gamma} - w_{\Gamma,n}|_{\mathbb{R}^m}\, d\Gamma \right) + \frac{\varepsilon^2}{2} \varliminf_{n\to\infty} \int_\Gamma \|\nabla_\Gamma w_{\Gamma,n}\|^2\, d\Gamma \\
&\geq \varliminf_{n\to\infty} |D[w_n]_{w_\Gamma}^{\mathrm{ex}}|(\overline{\Omega}) - \lim_{n\to\infty} \int_\Gamma |w_{\Gamma,n} - w_\Gamma|_{\mathbb{R}^m}\, d\Gamma + \frac{\varepsilon^2}{2} \int_\Gamma \|\nabla_\Gamma w_\Gamma\|^2\, d\Gamma \\
&\geq |D[w]_{w_\Gamma}^{\mathrm{ex}}|(\overline{\Omega}) + \frac{\varepsilon^2}{2}\int_\Gamma \|\nabla_\Gamma w_\Gamma\|^2\, d\Gamma = \Phi_*(W).
\end{aligned}$$

Thus, we conclude the Key-Lemma A. □

Next, we show the Key-Lemma B. This Key-Lemma can be obtained by means of a similar demonstration technique to that as in [10, Section 4]. Accordingly, we need to prepare the following this lemmas to prove Key-Lemma B.

Lemma B1. *Let $\mathbb{R}_+^N$ be the upper half-space of $\mathbb{R}^N$, i.e.:*

$$\mathbb{R}_+^N := \left\{ \, [\tilde{\xi},\xi_N] \in \mathbb{R}^N \;\middle|\; \tilde{\xi}\in\mathbb{R}^{N-1} \text{ and } \xi_N > 0 \,\right\}.$$

Then, for any $\varpi \in H^1(\mathbb{R}^{N-1};\mathbb{R}^m)\cap BV(\mathbb{R}^{N-1};\mathbb{R}^m)$, there exists a sequence $\{[\![\varpi]\!]_r^{\mathrm{ex}}\}_{r>0} \subset H^1(\mathbb{R}_+^N;\mathbb{R}^m)\cap BV(\mathbb{R}_+^N;\mathbb{R}^m)$, and for any $\tau>0$, there exists a small constant $r_\varpi^\tau \in (0,r_*]$, such that the following items hold.

$$\begin{gathered} r_\varpi^\tau \leq \tau \text{ and } [\![\varpi_r]\!]_r^{\mathrm{ex}}(\tilde{\xi},\xi_N) = 0, \text{ for any } r\in(0,r_\varpi^\tau] \\ \text{and a.e. } [\tilde{\xi},\xi_N]\in\mathbb{R}_+^N, \text{ satisfying } \xi_N > r; \end{gathered} \tag{3.9}$$

$$[\![\varpi]\!]_r^{\mathrm{ex}}{}_{|\mathbb{R}^{N-1}} = \varpi \text{ in } H^{\frac{1}{2}}(\mathbb{R}^{N-1};\mathbb{R}^m), \text{ for any } r\in(0,r_\varpi^\tau]; \tag{3.10}$$

$$\begin{gathered} |[\![\varpi]\!]_r^{\mathrm{ex}}|_{L^2(\mathbb{R}_+^N;\mathbb{R}^m)} \leq \tau, \text{ and } |D[\![\varpi]\!]_r^{\mathrm{ex}}|(\mathbb{R}_+^N) \leq |\varpi|_{L^1(\mathbb{R}^{N-1};\mathbb{R}^m)} + \tau, \\ \text{for any } r\in(0,r_\varpi^\tau]. \end{gathered} \tag{3.11}$$

Proof. For any $r>0$, and any function $\varpi\in H^1(\mathbb{R}^{N-1};\mathbb{R}^m)\cap BV(\mathbb{R}^{N-1};\mathbb{R}^m)$, we can define the sequence in the following form:

$$\begin{gathered} [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) = [\![\varpi]\!]_r^{\mathrm{ex}}(\tilde{\xi},\xi_N) := [1 - r^{-1}\xi_N]^+ \varpi(\tilde{\xi}), \\ \text{for a.e. } \tilde{\xi}\in\mathbb{R}^{N-1}, \text{ a.e. } \xi_N > 0 \text{ and any } r>0, \end{gathered} \tag{3.12}$$

and then, with [10] in mind, we can immediately check that $\{[\![\varpi]\!]_r^{\mathrm{ex}}\}_{r>0} \subset H^1(\mathbb{R}_+^N;\mathbb{R}^m)\cap BV(\mathbb{R}_+^N;\mathbb{R}^m)$. So, for any $\tau>0$, let us take a small constant $r_\varpi^\tau\in(0,\tau]$, such that:

$$r_\varpi^\tau \in (0,\tau], \ \sqrt{\frac{r_\varpi^\tau}{3}} \int_{\mathbb{R}^{N-1}} |\varpi|_{\mathbb{R}^m}\, d\tilde{\xi} < \tau \ \text{ and } \ \frac{r_\varpi^\tau}{2}\int_{\mathbb{R}^{N-1}} \|\tilde{\nabla}_\Gamma \varpi\|\, d\tilde{\xi} < \tau. \tag{3.13}$$

By means of (3.12)–(3.13), we can verify the condition (3.9). Also, let $\Phi := [\tilde{O}, \tilde{\varphi}] \in C^1(\overline{\mathbb{R}^N_+}; \mathbb{R}^{m\times N})$ be an arbitrary matrix-valued function with a zero matrix $\tilde{O} \in \mathbb{R}^{m\times(N-1)}$ and any m-dimensional vector $\tilde{\varphi} \in C^1(\mathbb{R}^{N-1}; \mathbb{R}^m)$, the condition (3.10) can be calculated as follows.

$$\begin{aligned}
&\int_{\mathbb{R}^{N-1}} \left([\![\varpi]\!]_r^{\mathrm{ex}}{}_{|\mathbb{R}^{N-1}} \cdot \tilde{\varphi} \right)(\tilde{\xi})\, d\tilde{\xi} = -\int_{\mathbb{R}^{N-1}} [\![\varpi]\!]_r^{\mathrm{ex}}{}_{|\mathbb{R}^{N-1}}(\tilde{\xi}) \cdot (\Phi e^N)(\tilde{\xi})\, d\tilde{\xi} \\
&= -\int_{\mathbb{R}^N_+} [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) \cdot \mathrm{div}\, \Phi(\xi)\, d\xi - \int_{\mathbb{R}^N_+} \nabla [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) : \Phi(\xi)\, d\xi \\
&= -\int_{\mathbb{R}^N_+} [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) \cdot (\partial_N \tilde{\varphi})(\tilde{\xi})\, d\xi \\
&\qquad - \int_{\mathbb{R}^N_+} \left[\tilde{\nabla} [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) : \tilde{O} + \left(\partial_N [\![\varpi]\!]_r^{\mathrm{ex}}\right)(\xi) \cdot \tilde{\varphi}(\tilde{\xi}) \right] d\xi \\
&= -\int_{\mathbb{R}^N_+} \left(\partial_N [\![\varpi]\!]_r^{\mathrm{ex}}\right)(\xi) \cdot \tilde{\varphi}(\tilde{\xi})\, d\xi = \int_{\mathbb{R}^{N-1}} (\varpi \cdot \tilde{\varphi})(\tilde{\xi})\, d\tilde{\xi}.
\end{aligned}$$

Additionally, with (3.12)–(3.13) in mind, we can compute that:

$$\begin{aligned}
\left| [\![\varpi]\!]_r^{\mathrm{ex}} \right|^2_{L^2(\mathbb{R}^N_+; \mathbb{R}^m)} &= \int_{\mathbb{R}^N_+} \left| [1 - r^{-1}\xi_N]^+ \varpi(\tilde{\xi}) \right|^2_{\mathbb{R}^m} d\xi \\
&= \left(\int_0^r (1 - r^{-1}\xi_N)^2\, d\xi_N \right) \left(\int_{\mathbb{R}^{N-1}} \left| \varpi(\tilde{\xi}) \right|^2_{\mathbb{R}^m} d\tilde{\xi} \right) \\
&= \frac{r}{3} \int_{\mathbb{R}^{N-1}} \left| \varpi(\tilde{\xi}) \right|^2_{\mathbb{R}^m} d\tilde{\xi} \leq \tau^2, \text{ for any } r \in (0, r^\tau_\varpi],
\end{aligned}$$

and

$$\begin{aligned}
&\left| D [\![\varpi]\!]_r^{\mathrm{ex}} \right| (\mathbb{R}^N_+) = \int_{\mathbb{R}^N_+} \left\| \nabla [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) \right\| d\xi \\
&\leq \int_{\mathbb{R}^N_+} \left\| \tilde{\nabla} [\![\varpi]\!]_r^{\mathrm{ex}}(\xi) \right\| d\xi + \int_{\mathbb{R}^N_+} \left| \left(\partial_N [\![\varpi]\!]_r^{\mathrm{ex}}\right)(\xi) \right|_{\mathbb{R}^m} d\xi \\
&= \int_{\mathbb{R}^N_+} \left\| [1 - r^{-1}\xi_N]^+ \tilde{\nabla} \varpi(\tilde{\xi}) \right\| d\xi + \int_{\mathbb{R}^N_+} \left| -r^{-1} \chi_{(0,r)}(\xi_N) \varpi(\tilde{\xi}) \right|_{\mathbb{R}^m} d\xi \\
&= \frac{r}{2} \int_{\mathbb{R}^{N-1}} \left\| \tilde{\nabla} \varpi \right\| d\tilde{\xi} + \int_{\mathbb{R}^{N-1}} \left| \varpi(\tilde{\xi}) \right|_{\mathbb{R}^m} d\tilde{\xi} \\
&= |\varpi|_{L^1(\mathbb{R}^{N-1}; \mathbb{R}^m)} + \tau, \text{ for any } r \in (0, r^\tau_\varpi].
\end{aligned}$$

Thus, we obtain Lemma B. □

Lemma B2. *For any $\hat{v} \in H^1(\Gamma; \mathbb{R}^m)$ and any $\ell \in \mathbb{N}$, there exists a function $\hat{v}_\ell \in H^1(\Omega; \mathbb{R}^m)$, satisfying $\hat{v}_\ell(x) = 0$, for a.e. $x \in \Omega \setminus \Gamma(2^{-\ell})$,*

$$\hat{v}_{\ell|\Gamma} = \hat{v}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m), \text{ and } |\hat{v}_\ell|_{L^2(\Omega; \mathbb{R}^m)} \leq 2^{-\ell}, \ |D\hat{v}_\ell|(\Omega) \leq |\hat{v}_\Gamma|_{L^1(\Gamma; \mathbb{R}^m)} + 2^{-\ell}.$$

Proof. Let $\sigma > 0$ be arbitrary, and let ρ_*^σ be the constant as in (Ω2). Then, just as in [10, Lemma 2], we can apply (Ω0)–(Ω1) to take:

- $m_\Omega^\sigma \in \mathbb{N}$, $\{x_{\Gamma,k}^\sigma\}_{k=1}^{m_\Omega^\sigma} \subset \Gamma$, and $G_k^\sigma := G_{x_{\Gamma,k}^\sigma}(\rho_*^\sigma, r_*)$, for all $k \in \{1, \dots, m_\Omega^\sigma\}$, as in ($\Omega$1), such that

$$\overline{\Gamma(r_*/2)} \subset G_*^\sigma := \bigcup_{k=1}^{m_\Omega^\sigma} G_k^\sigma; \tag{3.14}$$

- the partition of unity $\{\eta_k^\sigma\}_{k=1}^{m_\Omega^\sigma} \subset C_c^\infty(\mathbb{R}^N)$ for the covering G_*^σ, such that

$$0 \leq \eta_k^\sigma \in C_c^\infty(G_k^\sigma) \ \text{ for } k = 1, \dots, m_\Omega^\sigma, \text{ and } \sum_{k=1}^{m_\Omega^\sigma} \eta_k^\sigma = 1 \text{ on } \overline{\Gamma(r_*/2)}. \tag{3.15}$$

Next, for any $\tau > 0$, taking into account (Ω1) and Lemma B1, we put

$$\hat{r}_\sigma^\tau := \min\{r_{\varpi_k^\sigma}^\tau \,|\, k = 1, \dots, m_\Omega^\sigma\},$$

and define a function $\varpi_k^\sigma : \mathbb{R}^{N-1} \to \mathbb{R}^m$, as follows:

$$\varpi_k^\sigma(\tilde{\xi}) := \begin{cases} (\eta_k^\sigma \hat{v}_\Gamma)\big((\Xi_k^\sigma)^{-1}\tilde{\xi}\big), \\ \qquad \text{if } \tilde{\xi} \in \rho_*^\sigma \mathbb{B}^{N-1} \text{ and } k = 1, \dots, m_\Omega^\sigma, \quad \text{for a.e. } \tilde{\xi} \in \mathbb{R}^{N-1}, \\ 0, \quad \text{otherwise}, \end{cases} \tag{3.16}$$

where $\Xi_k^\sigma := \Xi_{x_{\Gamma,k}^\sigma}$ with $\Lambda_k^\sigma := \Lambda_{x_{\Gamma,k}^\sigma}$ and $H_k^\sigma := H_{x_{\Gamma,k}^\sigma}$, for all $k \in \{1, \dots, m_\Omega^\sigma\}$.

Based on these, we define a class of functions $\{\hat{v}_\sigma^\tau \,|\, \sigma, \tau > 0\}$, as follows:

$$\hat{v}_\sigma^\tau(x) := \begin{cases} \displaystyle\sum_{k=1}^{m_\Omega^\sigma} [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\Xi_k^\sigma x), \\ \qquad \text{if } x \in G_k^\sigma, \text{ for some } k \in \{1, \dots, m_\Omega^\sigma\}, \\ 0, \quad \text{otherwise}, \end{cases} \tag{3.17}$$
$$\text{for a.e. } x \in \Omega \text{ and all } \sigma, \tau > 0.$$

Then, as direct consequences of (3.14)–(3.17) and Lemma B1, it is inferred that:

$$\begin{aligned} &\hat{v}_\sigma^\tau \in H^1(\Omega; \mathbb{R}^m), \ \hat{v}_{\sigma|\Gamma}^\tau = \hat{v}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma; \mathbb{R}^m), \\ &\text{and } \hat{v}_\sigma^\tau = 0 \text{ a.e. on } \Omega \setminus \Gamma(\tau), \text{ for all } \sigma, \tau > 0. \end{aligned} \tag{3.18}$$

Also, in the light of (3.11), (Ω2) and Lemma B1, we compute that:

$$\begin{aligned} |\hat{v}_\sigma^\tau|_{L^2(\Omega;\mathbb{R}^m)} &= \left[\int_\Omega \left|\sum_{k=1}^{m_\Omega^\sigma} [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\Xi_k^\sigma x)\right|^2 dx\right]^{\frac{1}{2}} \leq \sum_{k=1}^{m_\Omega^\sigma} \left[\int_{\mathbb{R}_+^N} \left|[\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\xi)\right|^2 d\xi\right]^{\frac{1}{2}} \\ &\leq m_\Omega^\sigma \tau, \text{ for all } \sigma, \tau > 0, \end{aligned} \tag{3.19}$$

and

$$
\begin{aligned}
\int_\Omega \|\nabla_x \hat{v}_\sigma^\tau(x)\|\,dx &= \int_\Omega \Big\|\sum_{k=1}^{m_\Omega^\sigma} \nabla_x [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\Xi_k^\sigma x)\Big\|\,dx \\
&\leq \sum_{k=1}^{m_\Omega^\sigma} \int_{G_k^\sigma \cap \Omega} \big\|\nabla_x [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\Xi_k^\sigma x)\big\|\,dx \\
&= \sum_{k=1}^{m_\Omega^\sigma} \int_{Y_k^\sigma \cap (\Lambda_k^\sigma \Omega)} \big\|\nabla_y [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(H_k^\sigma y)\big\|\,dy \\
&\leq \sum_{k=1}^{m_\Omega^\sigma} \left(\int_{\mathbb{R}_+^N} \big\|\nabla_\xi [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\xi)\big\|\,d\xi + \int_{\mathbb{R}_+^N} \big\|\tilde{\nabla}\gamma_{x_\Gamma}(\tilde{\xi})\big\| \big|(\partial_{\xi_N} [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}})(\xi)\big|_{\mathbb{R}^m}\,d\xi \right) \\
&\leq \sum_{k=1}^{m_\Omega^\sigma} \left(\int_{\mathbb{R}_+^N} \big\|\nabla_\xi [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\xi)\big\|\,d\xi + \big|\tilde{\nabla}\gamma_{x_\Gamma}\big|_{C(\rho_*^\sigma \overline{\mathbb{B}^{N-1}})} \int_{\mathbb{R}_+^N} \big|(\partial_{\xi_N} [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}})(\xi)\big|_{\mathbb{R}^m}\,d\xi \right) \\
&\leq \left(1 + |\nabla\gamma_{x_\Gamma}|_{C(\rho_*^\sigma \overline{\mathbb{B}^{N-1}})}\right) \sum_{k=1}^{m_\Omega^\sigma} \int_{\mathbb{R}_+^N} \big\|\nabla_\xi [\![\varpi_k^\sigma]\!]_{\hat{r}_\sigma^\tau}^{\mathrm{ex}}(\xi)\big\|\,d\xi \\
&\leq (1+\sigma) \sum_{k=1}^{m_\Omega^\sigma} \left(\int_{\mathbb{R}^{N-1}} |\varpi_k^\sigma(\tilde{\xi})|_{\mathbb{R}^m}\,d\tilde{\xi} + \tau \right) \\
&\leq (1+\sigma) \sum_{k=1}^{m_\Omega^\sigma} \left(\int_{G_k^\sigma \cap \Gamma} \eta_k^\sigma |\hat{v}_\Gamma|_{\mathbb{R}^m}\,d\Gamma + \tau \right) \\
&\leq (1+\sigma)|\hat{v}_\Gamma|_{L^1(\Gamma;\mathbb{R}^m)} + m_\Omega^\sigma \tau (1+\sigma), \text{ for all } \sigma, \tau > 0.
\end{aligned}
\tag{3.20}
$$

Now, for any $\ell \in \mathbb{N}$, let us take two constants $\sigma_\ell, \tau_\ell \in (0,1]$, such that:

$$
\left\{
\begin{array}{l}
(1+\sigma_\ell)|\hat{v}_\Gamma|_{L^1(\Gamma;\mathbb{R}^m)} \leq |\hat{v}_\Gamma|_{L^1(\Gamma;\mathbb{R}^m)} + 2^{-\ell-1}, \\[2mm]
\tau_\ell + m_\Omega^{\sigma_\ell}\tau_\ell(1+\sigma_\ell) \leq 2^{-\ell-1},
\end{array}
\right.
\quad \text{for } \ell \in \mathbb{N}.
\tag{3.21}
$$

Then, on account of (3.18)–(3.21), we will conclude that the function $\hat{v}_\ell := \hat{v}_{\sigma_\ell}^{\tau_\ell} \in H^1(\Omega;\mathbb{R}^m)$, for each $\ell \in \mathbb{N}$, will fulfill the required condition. □

Based on these, the Key-Lemma B is demonstrated as follows.

Proof of Key-Lemma B. The proof of Key-Lemma B is a modified version of [7, Theorem 6] and [10, key-Lemma A]. For any $\hat{w} \in BV(\Omega;\mathbb{R}^m) \cap L^2(\Omega;\mathbb{R}^m)$, we can find a sequence $\{\hat{\varphi}_\ell\}_{\ell=1}^\infty \subset C^\infty(\overline{\Omega};\mathbb{R}^m)$, such that:

$$
|\hat{\varphi}_\ell - \hat{w}|_{L^2(\Omega;\mathbb{R}^m)} \leq 2^{-\ell-1} \text{ and } \left| \int_\Omega \|\nabla\hat{\varphi}_\ell\|\,dx - \int_\Omega |D\hat{w}| \right| \leq 2^{-\ell-1}, \text{ for any } \ell \in \mathbb{N},
$$

and from Remark 1.2, we can say that:

$$
\hat{\varphi}_{\ell|_\Gamma} \to \hat{w}_{|_\Gamma} \text{ in } L^1(\Gamma;\mathbb{R}^m), \text{ as } \ell \to \infty.
$$

Next, we apply Lemma B2 as the case when $\hat{v}_\Gamma := \hat{w}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}$ in $H^{\frac{1}{2}}(\Gamma;\mathbb{R}^m)$. Then, for any $\ell \in \mathbb{N}$, we can take a function $\hat{\psi}_\ell \in H^1(\Omega;\mathbb{R}^m)$, such that:

$$\hat{\psi}_{\ell|_\Gamma} = \hat{v}_\Gamma = \hat{w}_\Gamma - \hat{\varphi}_{\ell|_\Gamma} \text{ in } H^{\frac{1}{2}}(\Gamma;\mathbb{R}^m),$$

$$|\hat{\psi}_\ell|_{L^2(\Omega;\mathbb{R}^m)} \leq 2^{-\ell-1} \text{ and } |D\hat{\psi}_\ell|(\Omega) \leq |\hat{w}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}|_{L^1(\Gamma;\mathbb{R}^m)} + 2^{-\ell-1}. \tag{3.22}$$

Now, let us define:

$$\hat{w}_\ell := \hat{\varphi}_\ell + \hat{\psi}_\ell \text{ in } L^2(\Omega;\mathbb{R}^m), \text{ for any } \ell \in \mathbb{N}. \tag{3.23}$$

Then, one can easily check that:

$$\hat{w}_{\ell|_\Gamma} = \hat{\varphi}_{\ell|_\Gamma} + \hat{\psi}_{\ell|_\Gamma} = \hat{w}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma;\mathbb{R}^m), \text{ for any } \ell \in \mathbb{N}, \tag{3.24}$$

and

$$|\hat{w}_\ell - \hat{w}|_{L^2(\Omega;\mathbb{R}^m)} \leq |\hat{\varphi}_\ell - \hat{w}|_{L^2(\Omega;\mathbb{R}^m)} + |\hat{\psi}_\ell|_{L^2(\Omega;\mathbb{R}^m)} \leq 2^{-\ell}, \text{ for any } \ell \in \mathbb{N}. \tag{3.25}$$

Also, with (3.22) in mind, we can complete that:

$$\begin{aligned}
&\int_\Omega \|\nabla \hat{w}_\ell\|\,dx + \int_\Gamma |\hat{w}_{\ell|_\Gamma} - \hat{w}_\Gamma|_{\mathbb{R}^m}\,d\Gamma \\
&\leq \int_\Omega \|\nabla \hat{\varphi}_\ell\|\,dx + \int_\Omega \|\nabla \hat{\psi}_\ell\|\,dx \\
&\leq \int_\Omega \|\nabla \hat{\varphi}_\ell\|\,dx + \int_\Gamma |\hat{w}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}|_{\mathbb{R}^m}\,d\Gamma + 2^{-\ell}, \text{ for any } \ell \in \mathbb{N}.
\end{aligned} \tag{3.26}$$

Furthermore, on account of (3.23)–(3.26) and Key-Lemma A, it is deduced that:

$$\begin{aligned}
&\int_\Omega |D\hat{w}| + \int_\Gamma |\hat{w}_{|_\Gamma} - \hat{w}_\Gamma|_{\mathbb{R}^m}\,d\Gamma \\
&\leq \varliminf_{\ell\to\infty} \left(\int_\Omega \|\nabla \hat{w}_\ell\|\,dx + \int_\Gamma |\hat{w}_{\ell|_\Gamma} - \hat{w}_\Gamma|_{\mathbb{R}^m}\,d\Gamma \right) \\
&\leq \varlimsup_{\ell\to\infty} \int_\Omega \|\nabla \hat{\varphi}_\ell\|\,dx + \varlimsup_{\ell\to\infty} \left(\int_\Gamma |\hat{w}_\Gamma - \hat{\varphi}_{\ell|_\Gamma}|_{\mathbb{R}^m}\,d\Gamma + 2^{-\ell} \right) \\
&= \int_\Omega |D\hat{w}| + \int_\Gamma |\hat{w}_{|_\Gamma} - \hat{w}_\Gamma|_{\mathbb{R}^m}\,d\Gamma.
\end{aligned}$$

Thus, we conclude the Key-Lemma B. □

4 Proof of Main Theorem

This section is devoted to the proof of the Main Theorem 1.

Proof of Main Theorem 1. First, we verify the condition of lower-bound. So, we assume that under (2.1):

$$\check{U}_n \to \check{U} \text{ weakly in } \mathscr{H}, \text{ as } n \to \infty, \tag{4.1}$$

for any $\check{U} := [\check{u}, \check{u}_\Gamma] \in \mathscr{H}$, and any sequence $\{\check{U}_n := [\check{u}_n, \check{u}_{\Gamma,n}]\}_{n=1}^{\infty} \subset \mathscr{H}$. Then, under the assumption (4.1), we may suppose the presence of a subsequence $\{k\} \subset \{n\} \subset \mathbb{N}$ and a constant $\check{\Phi} \in [0, \infty)$, such that:

$$\check{\Phi} := \varliminf_{n\to\infty} \Phi_n(\check{U}_n) = \lim_{k\to\infty} \Phi_k(\check{U}_k) < \infty, \tag{4.2}$$

because the other cases are trivial. Additionally, under (4.2), we can say that:

$$\begin{cases} \{\check{U}_k\}_{k=1}^{\infty} \subset \mathscr{V}, \text{ therefore } \check{u}_{k|\Gamma} = \check{u}_{\Gamma,k} \text{ on } \Gamma, \\ \{\check{U}_k\}_{k=1}^{\infty} \text{ is bounded in } \mathscr{W} := (BV(\Omega;\mathbb{R}^m) \cap L^2(\Omega;\mathbb{R}^m)) \times H^1(\Gamma;\mathbb{R}^m), \end{cases} \tag{4.3}$$

and

$$\begin{cases} \check{u}_k \to \check{u} \text{ in } L^1(\Omega;\mathbb{R}^m) \text{ and weakly in } L^2(\Omega;\mathbb{R}^m), \\ \check{u}_{\Gamma,k} \to \check{u}_\Gamma \text{ weakly in } H^1(\Gamma;\mathbb{R}^m), \end{cases} \quad \text{as } k \to \infty, \tag{4.4}$$

by taking more subsequence if necessary.

On account of (4.2)–(4.4), the assumption (a1) and Key-Lemma A, we can compute that:

$$\begin{aligned} \check{\Phi} &= \varliminf_{n\to\infty} \Phi_n(\check{U}_n) = \lim_{k\to\infty} \Phi_k(\check{U}) \\ &\geq \varliminf_{k\to\infty} \int_\Omega f_{\delta_k}(\nabla \check{u}_k)\,dx + \frac{1}{2} \varliminf_{k\to\infty} \int_\Omega \|\nabla(\kappa_k \check{u}_k)\|^2\,dx + \frac{\varepsilon^2}{2} \varliminf_{k\to\infty} \int_\Gamma \|\nabla_\Gamma \check{u}_k\|^2\,d\Gamma \\ &\geq \varliminf_{k\to\infty} \int_\Omega (\|\nabla \check{u}_k\| - \delta_k C_0)\,dx + \varliminf_{k\to\infty} \int_\Gamma |\check{u}_{k|\Gamma} - \check{u}_{\Gamma,k}|_{\mathbb{R}^m}\,d\Gamma + \frac{\varepsilon^2}{2} \varliminf_{k\to\infty} \int_\Gamma \|\nabla_\Gamma \check{u}_k\|^2\,d\Gamma \\ &\geq \varliminf_{k\to\infty} \Phi_*(\check{U}_k) \geq \Phi_*(\check{U}). \end{aligned}$$

Thus, we verify the condition of lower-bound.

Next, we verify the condition of optimality. Let us fix any function $\hat{U} = [\hat{u}, \hat{u}_\Gamma] \in \mathscr{W}$. Then, Key-Lemma B enables us to take a sequence $\{\hat{V}_\ell = [\hat{v}_\ell, \hat{v}_{\Gamma,\ell}]\} \subset \mathscr{V}$ such that:

$$\hat{v}_{\Gamma,\ell} = \hat{v}_{\ell|\Gamma} = \hat{u}_\Gamma \text{ in } H^{\frac{1}{2}}(\Gamma;\mathbb{R}^m), \text{ for any } \ell \in \mathbb{N}, \tag{4.5}$$

and

$$\begin{cases} |\hat{v}_\ell - \hat{u}|_{L^2(\Omega;\mathbb{R}^m)} < 2^{-\ell}, \\ \left| \displaystyle\int_\Omega \|\nabla \hat{v}_\ell\|\,dx - \left(\int_\Omega |D\hat{u}| + \int_\Gamma |\hat{u}_{|\Gamma} - \hat{u}_\Gamma|_{\mathbb{R}^m}\,d\Gamma \right) \right| < 2^{-\ell-2}, \end{cases} \quad \text{for any } \ell \in \mathbb{N}. \tag{4.6}$$

In the meantime, by the assumption (A1), we have

$$0 \leq f_\delta(\nabla \hat{v}_\ell) \leq \nabla f_\delta(\nabla \hat{v}_\ell) : \nabla \hat{v}_\ell \leq C_1 \|\nabla \hat{v}_\ell\|^2 + C_2 \|\nabla \hat{v}_\ell\|, \text{ for any } \ell \in \mathbb{N}. \tag{4.7}$$

Then, with (4.6)–(4.7) and the assumption (a2) in mind, we can apply Lebesgue's dominated convergence Theorem, and can configure a large number $n_\ell \in \mathbb{N}$ such that:

$$\begin{cases} \left(\displaystyle\sup_{n \geq n_\ell} \frac{\kappa_n^2}{2} \right) \left(\int_\Omega \|\nabla \hat{v}_\ell\|^2\,dx \right) < 2^{-\ell-2}, \\ \displaystyle\sup_{n \geq n_\ell} \left| \int_\Omega f_{\delta_n}(\nabla \hat{v}_\ell)\,dx - \int_\Omega \|\nabla \hat{v}_\ell\|\,dx \right| < 2^{-\ell-2}, \end{cases} \quad \text{for any } \ell \in \mathbb{N}. \tag{4.8}$$

Now, we define a sequence $\{\hat{U}_n = [\hat{u}_n, \hat{u}_{\Gamma,n}]\}_{n=1}^{\infty} \subset \mathscr{V}$, by putting:

$$\hat{U}_n = [\hat{u}_n, \hat{u}_{\Gamma,n}] := \begin{cases} [\hat{v}_\ell, \hat{v}_{\Gamma,\ell}] \text{ in } \mathscr{V}, \\ \qquad \text{if } n_\ell \leq n < n_{\ell+1}, \text{ for some } \ell \in \mathbb{N}, \\ [\hat{v}_1, \hat{v}_{\Gamma,1}] \text{ in } \mathscr{V}, \\ \qquad \text{if } 1 \leq n < n_1, \end{cases} \quad \text{for } n = 1, 2, 3, \ldots. \tag{4.9}$$

Then, in the light of (4.5)–(4.6), (4.8)–(4.9), it is inferred that:

$$\begin{gathered} |\hat{U}_n - \hat{U}|_{\mathscr{H}} = |\hat{v}_\ell - \hat{u}|_{L^2(\Omega;\mathbb{R}^m)} + |\hat{v}_{\Gamma,\ell} - \hat{u}_\Gamma|_{L^2(\Gamma;\mathbb{R}^m)} < 2^{-\ell}, \\ \text{for any } n \geq n_\ell \text{ and some } \ell \in \mathbb{N}, \end{gathered} \tag{4.10}$$

and

$$\begin{aligned} &|\Phi_n(\hat{U}_n) - \Phi_*(\hat{U})| \\ &\leq \left| \int_\Omega \left(f_{\delta_n}(\nabla \hat{u}_n) + \frac{\kappa_n^2}{2} \|\nabla \hat{u}_n\|^2 \right) dx - \left(\int_\Omega |D\hat{u}| + \int_\Gamma |\hat{u}_{|\Gamma} - \hat{u}_\Gamma|_{\mathbb{R}^m} \, d\Gamma \right) \right| \\ &\qquad + \frac{\varepsilon^2}{2} \left| \int_\Gamma (\|\nabla_\Gamma \hat{u}_{\Gamma,n}\|^2 - \|\nabla_\Gamma \hat{u}_\Gamma\|^2) \, d\Gamma \right| \\ &\leq \left| \int_\Omega f_{\delta_n}(\nabla \hat{u}_n) \, dx - \int_\Omega \|\nabla \hat{u}_n\| \, dx \right| + \frac{\kappa_n^2}{2} \int_\Omega \|\nabla \hat{u}_n\|^2 \, dx \\ &\qquad + \left| \int_\Omega \|\nabla \hat{u}_n\| \, dx - \left(\int_\Omega |D\hat{u}| + \int_\Gamma |\hat{u}_{|\Gamma} - \hat{u}_\Gamma|_{\mathbb{R}^m} \, d\Gamma \right) \right| \\ &< 2^{-\ell}, \text{ for any } n_\ell \leq n < n_{\ell+1}, \text{ and any } \ell \in \mathbb{N}. \end{aligned} \tag{4.11}$$

The above calculations (4.9)–(4.11) imply that:

$$\hat{u}_n \to \hat{u} \text{ in } L^2(\Omega; \mathbb{R}^m), \text{ and } \Phi_n(\hat{U}_n) \to \Phi_*(\hat{U}) \text{ as } n \to \infty,$$

required in the condition of optimality.

Thus, we conclude the Main Theorem 1. □

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